

KINETICS OF SIMILARITY REGIME OF VAPORIZATION
OF A POLYDISPERSE SYSTEM OF DROPS

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A method is proposed for calculating heat and mass transfer between phases on the basis of analysis of a kinetic equation for a drop-radius distribution function.

The processes involved in the interaction of a gas with vaporizing drops are put to use in many engineering devices — in spray drying, in evaporating cooling systems, etc. These devices are very efficient while remaining structurally simple [1]. Their operating regimes are intimately related to the kinetics of heat and mass transfer between the phases. Questions concerning the calculation of such transfer processes have been addressed in monographs and numerous journal articles. The greatest difficulties arise in examining polydisperse drop systems. Numerical methods [1, 2] are generally used in this case. Mass transfer in a polydisperse system of particles was described analytically in [3, 4] on the basis of a kinetic equation.

This approach is used below to analyze the vaporization of a polydisperse system of drops in a vapor-gas mixture and its cooling. As in [2-4], we assume that the drops are spherical and vaporize independently of each other, that the pressure of the saturated vapor is independent of the drop radius, and that no comminution or coalescence of the drops occurs. Also, we will regard the gas and drops as an adiabatic system consisting of two subsystems in local equilibrium. Here, because of the rapid thermal relaxation of the drops, their temperature may be assumed constant and equal to the temperature of a wet thermometer. The foregoing assumptions are realized in many cases of practical importance.

We will describe the change in the polydispersity of the particles during vaporization with a kinetic equation for the distribution function of particle radius r :

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial r} (\omega f) = 0. \quad (1)$$

The following relations [4] are satisfied for $f(r, t)$:

$$dN = N_0 f(r, t) dr, \quad f(r, 0) = f_0(r), \quad \int_0^{\infty} f_0(r) dr = 1. \quad (2)$$

These relations must be supplemented by the independent equation of the kinetics of vaporization of an individual drop $dr/dt = w(r, t)$. In the general case of an arbitrary dependence of vaporization rate $w(r, t)$ on radius r , integration of Eq. (1) is a fairly complicated task in view of its nonlinearity. Methods of rigorously solving the equation were developed in detail in [4], but a numerical calculation is necessary to obtain specific results. The problem can be simplified considerably for similarity regimes of vaporization, when the parameters of the system remain constant with a change in the number of drops per unit volume of the system. Such regimes may prevail, for example, in the intensive vaporization of drops in a highly turbulent gas [4]. The vaporization of individual drops is usually regarded as a quasisteady process [1-5]. Here, the vaporization rate $w(r, t)$ can very often be represented in the form of the product of two functions $w = \omega(t)\Omega(r)$. This makes it possible to use the common procedure of separation of variables by Fourier's method. Assuming in (1) that $f(r, t) = \Phi(t)R(r)$, we obtain the system of ordinary differential equations

$$\frac{d\Phi}{dt} = \alpha\omega\Phi, \quad \frac{d(\Omega R)}{dr} = -aR, \quad (3)$$

the solution of which may be represented in the form

$$\Phi = \exp \left[a \int_0^t \omega(t) dt \right], \quad R(r) = A \Omega^{-1} \exp \left[-a \int_0^t \Omega^{-1}(r) dr \right]. \quad (4)$$

Thus, the general solution of (1) may be written in the form of a series

$$f(r, t) = \sum_i A_i \Omega^{-1}(r) \exp \left[-a_i \int \Omega^{-1}(r) dr \right] \exp \left[a_i \int_0^t \omega(t) dt \right]. \quad (5)$$

In many cases of practical importance, the vaporization kinetics may be conveniently given through the heat-balance equation for individual drops

$$\frac{dr}{dt} = - \frac{\alpha(r)}{\rho_q r_i} (T - T_q) = - \frac{\alpha(r)}{\rho_q r_i} \vartheta. \quad (6)$$

If necessary, Eq. (6) may be replaced by other expressions for w which describe drop vaporization with more detail and rigor, such as in [6]. However, all of these expressions are generally cumbersome and involve the use of numerical methods to achieve the final results. Moreover, (6) allows the use of empirical data on heat transfer for engineering calculations. Based on this data, α may be represented in the form $\alpha = \eta r^{-n}$ [1, 2, 4, 5, 7]. The value of η depends on the rate of flow and thermophysical properties of the medium. At $n=1$, Eq. (6) gives the familiar Maxwell vaporization law [5]. The case $n=0$ corresponds to vaporization in a highly turbulent gas [4]. The heat-transfer coefficient may be calculated on the basis of any determining dimension convenient for practical purposes. For example, if the initial mean particle radius \bar{r}_0 is chosen as this dimension, then $\alpha_0 = \eta \bar{r}_0^{-1}$ and $\alpha = \alpha_0 (r/\bar{r}_0)^{-n}$. Thus, the vaporization law for an individual particle will have the form

$$\omega = \frac{dr}{dt} = \left(\frac{r}{\bar{r}_0} \right)^{-n} \frac{\alpha_0 \vartheta(t)}{\rho_q r_i}, \quad (7)$$

where

$$(r/\bar{r}_0)^{-n} = \Omega(r); \quad -\alpha_0/\rho_q r_i = \omega(t).$$

The integral of r in (5) is easily calculated:

$$\int \Omega^{-1}(r) dr = \int \left(\frac{r}{\bar{r}_0} \right)^n dr = \frac{r^{n+1}}{(n+1) \bar{r}_0^n}, \quad (8)$$

and, allowing for (7) and (8), we may represent distribution function (5) as follows:

$$f(r, t) = \sum_i A_i \left(\frac{r}{\bar{r}_0} \right)^n \exp \left[-\frac{a_i r^{n+1}}{(n+1) \bar{r}_0^n} \right] \exp \left[-a_i^* \int_0^t v(t) dt \right], \quad (9)$$

where $a_i^* = a \alpha_0 / \rho_q r_i$.

At $t=0$ from (9) we obtain an approximation of the initial distribution

$$f_0(r) = f(r, 0) = \sum_i A_i \left(\frac{r}{\bar{r}_0} \right)^n \exp \left[-\frac{a_i r^{n+1}}{(n+1) \bar{r}_0^n} \right]. \quad (10)$$

Using (9), we calculate the moment of order s :

$$\langle r^s(t) \rangle = \frac{N_0}{N(t)} \int_0^\infty r^s f(r, t) dr = \frac{N_0}{N} \sum_i A_i a_i^{-\frac{n+s+1}{n+1}} \exp \left[-a_i^* \int_0^t \vartheta dt \right]. \quad (11)$$

In the integration of $f(r, t)$ for r in (11), it was considered in [8] that

$$\int_0^\infty \frac{r^{n+s}}{\bar{r}_0^n} \exp \left[-\frac{a_i r^{n+1}}{(n+1) \bar{r}_0^n} \right] dr = C_s a_i^{-\frac{n+s+1}{n+1}} \bar{r}_0^{\frac{ns}{n+1}}, \quad (12)$$

$$C_s = (n+1)^{\frac{s}{n+1}} \Gamma\left(\frac{n+s+1}{n+1}\right). \quad (13)$$

For subsequent calculations, it would be convenient to introduce the quantity $y=1-x$, representing the ratio of the unvaporized mass of drops $M_q(t)$ to the initial mass of the drops M_{q0} :

$$y(t) = \frac{M_q}{M_{q0}} = \frac{1}{r_0^3} \int_0^\infty r^3 f(r, t) dr = \frac{\langle r^3 \rangle N(t)}{r_0^3 N_0}. \quad (14)$$

Substituting (11) into (14), for $s=3$ we find that

$$y(t) = \frac{C_3}{r_0^3} \frac{3n}{r_0^{\frac{3n}{n+1}}} \sum_i A_i a_i^{-\frac{n+4}{n+1}} \exp\left[-a_i^* \int_0^t \vartheta(t) dt\right]. \quad (15)$$

Beginning with a certain moment of time, the behavior of functions (9), (11), and (15) will be determined by the term with the lowest a_i ($\min\{a_i\}=a$), i.e., a unique regular vaporization regime will begin. In this case, the initial distribution (10) will be approximated by one of the functions belonging to the class $R(r)$ [4].

Let the regular regime begin at moment of time t_0 . Then, for $t \geq t_0$, we may ignore all of the terms of series (11) except one:

$$\langle r^s \rangle = \frac{C_s N_0 \bar{r}_0^{\frac{ns}{n+1}}}{N(t)} A a^{-\frac{n+s+1}{n+1}} \exp\left[-a^* \int_{t_0}^t \vartheta(t) dt\right]. \quad (16)$$

Assuming $s=0$ in (16) and considering from (13) that $C_s=C_0=1$, we obtain

$$N(t) = v N_0 \exp\left[-a^* \int_{t_0}^t \vartheta(t) dt\right], \quad v = \frac{A}{a} = \frac{N(t_0)}{N_0}. \quad (17)$$

We find the following similarity parameters from (11) and (17)

$$\langle r^s \rangle = \bar{r}_0^s = v C_s a^{-\frac{s}{n+1}} \bar{r}_0^{\frac{ns}{n+1}}. \quad (18)$$

Determining a from (18) at $s=1$, we finally obtain

$$a = \frac{(v C_1)^{\frac{n+1}{n}}}{r_0}, \quad \langle r^s \rangle = \bar{r}_0^s = C_s \left(\frac{\bar{r}_0}{C_1 v}\right)^s. \quad (19)$$

Allowing for (14), (17), and (18), Eq. (15) takes the following form for the regular regime

$$y = v \exp\left[-a^* \int_{t_0}^t \vartheta(t) dt\right]. \quad (20)$$

Differentiating (20) with respect to t , we change over to the equation

$$\frac{dy}{dt} + a^* \vartheta y = 0. \quad (21)$$

We will write the heat balance of the system thus

$$\frac{dQ_q}{dt} - (M_g c_{pg} + M_v c_{pv}) \frac{d\vartheta}{dt} = 4\pi N_0 \alpha_0 \vartheta \int_0^\infty r^2 \left(\frac{r}{r_0}\right)^{-n} f(r, t) dt. \quad (22)$$

The integral on the right side can be expressed through moments (11) and (19) of the order $s=2-n$

$$\int_0^\infty r^2 \left(\frac{r}{r_0}\right)^{-n} f(r, t) dr = \frac{N(t)}{N_0} \bar{r}_0^n \langle r^{2-n} \rangle. \quad (23)$$

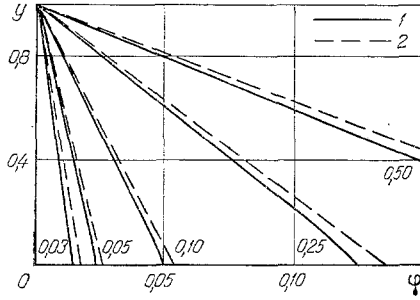


Fig. 1

Fig. 1. Curve of the function $y = y(\varphi)$; 1) calculated with exact Eq. (27); 2) linear approximation (29). The numbers denote values of μ .

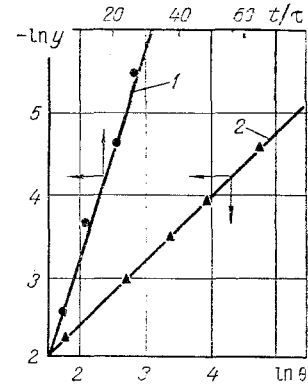


Fig. 2

Fig. 2. Kinetics of vaporization of a drop system: 1) calculated with (22); 2) with (23). The corresponding points denote data from [2] for $\vartheta^*/\vartheta_0 = 0.8$ and 1.0.

Considering that $M_{q0} = \frac{4}{3} \pi r_0^2 \rho_q$ and taking into account the definition of a^* in (9) and Eqs. (13) and (19), we introduce the complexes:

$$\mu = \frac{M_{q0}}{M_g}, \quad a^* = \frac{\alpha_0 (vC_1)^{n+1}}{\rho_q r_i r_0}, \quad \vartheta^* = \frac{3\mu r_i v}{c_i g k_r}, \quad (24)$$

$$k_r = \frac{C_3}{C_{2-n}} = \Gamma \left(\frac{n+4}{n+1} \right) \left[\Gamma \left(\frac{3}{n+1} \right) \right]^{-1} (n+1)^{-1}.$$

Equation (22) is changed to the following form by means of (23) and (24)

$$\left[1 + \mu (1-y) \frac{c_{rv}}{c_{pg}} \right] \frac{d\vartheta}{dt} = -a^* \vartheta^* \vartheta \exp \left[-a^* \int_{\vartheta_0}^{\vartheta} \vartheta(t) dt \right]. \quad (25)$$

Examining (21) and (25) together, we may exclude the time t , having determined $y = y(\vartheta)$:

$$\vartheta^* \frac{dy}{d\vartheta} = \left[1 + \frac{\mu c_{rv}}{c_{pg}} (1-y) \right]. \quad (26)$$

For the initial conditions $\vartheta|_{t=t_0} = \vartheta_0 = T_0 - T_q$, $y(\vartheta_0) = 1$, the solution of (26) has the form

$$y = 1 - \frac{c_{pg}}{\mu c_{rv}} \left\{ \exp \left[\frac{\mu c_{rv}}{c_{pg} \vartheta^*} (\vartheta_0 - \vartheta) \right] - 1 \right\}. \quad (27)$$

From here we find ϑ as a function of y and substitute $\vartheta(y)$ into (21):

$$\frac{dy}{dt} + a^* \left[\vartheta_0 - \frac{c_{pg} \vartheta^*}{\mu c_{rv}} \ln \left| 1 + \frac{\mu c_{rv}}{c_{pg}} (1-y) \right| \right] y = 0. \quad (28)$$

This equation reduces to quadratures, but its solution is not expressed in terms of elementary functions and involves the use of numerical methods. Thus, (26) and (27) fully describe the stated problem.

If we linearize (27)

$$y = 1 - \frac{\vartheta_0 - \vartheta}{\vartheta^*}, \quad (29)$$

then (28) takes the form

$$\frac{dy}{dt} + a^* (\vartheta_0 - \vartheta^*) y + a^* \vartheta^* y^2 = 0. \quad (30)$$

It coincides with the equation in [4] for the time part of the distribution function. Numerical calculations of $y = y(\varphi)$, $\varphi = \mu c_{pv}(\vartheta_0 - \vartheta) / c_{pg}^*$ using exact and approximate formulas (16) and (18), shown in Fig. 1, indicate that, in cases of practical importance ($\mu \leq 0.5$) [1], it is quite acceptable to limit oneself to the linear approximation (29) for $y = y(\vartheta)$. The function $y(t)$ is easily found by integrating (30). With the initial condition $y(0) = 1$, the solution of (30) has the form

$$y = (\vartheta_0 - \vartheta^*) \{ \vartheta_0 \exp [a^* (\vartheta_0 - \vartheta^*) t] - \vartheta^* \}^{-1}. \quad (31)$$

The quantity of moisture vaporized up to moment of time t is equal to

$$x = 1 - y = \frac{\exp [a^* (\vartheta_0 - \vartheta^*) t] - 1}{\exp [a^* (\vartheta_0 - \vartheta^*) t] - \vartheta^* / \vartheta_0}.$$

The temperature of the gas is determined as a function of time from (27)

$$T(t) = T_0 - \frac{c_{pg} \vartheta^*}{\mu c_{pv}} \ln \left| 1 + \frac{\mu c_{pv} \{ \exp [a^* (\vartheta_0 - \vartheta^*) t] - 1 \}}{c_{pg} \{ \exp [a^* (\vartheta_0 - \vartheta^*) t] - \vartheta^* / \vartheta_0 \}} \right|. \quad (32)$$

The resulting relations include the following special cases. If $\vartheta_0 > \vartheta^*$, then, for time $t \gg [a^* (\vartheta_0 - \vartheta^*)]^{-1} = \tau$ from (31) we have

$$y = (1 - \vartheta^* / \vartheta_0) \exp (-t / \tau). \quad (33)$$

If $\vartheta_0 = \vartheta^*$, the air will be cooled to the temperature of the drops T_Q . Here, all of the sprayed moisture will be vaporized by the end of the process and the solution of (30) will have the form

$$y = (1 + \theta)^{-1}, \quad \theta = a^* \vartheta_0 t = \frac{\alpha_0 (vC_1)^{n+1} (T_0 - T_Q) t}{\rho_Q r_0}. \quad (34)$$

At $\vartheta_0 < \vartheta^*$, some of the moisture will remain unvaporized

$$y_v = 1 - \frac{c_{pg} \vartheta_0 k_r}{3 \mu r_{iv}}.$$

The results of the calculations with Eqs. (33) and (34) (Fig. 2) for time $t > \tau$, $\theta > 1$ agree well with the data in [2]. If we use (30) to calculate the change in temperature over time, then integral equation (25) might lead to the nonlinear differential equation

$$\vartheta \frac{d^2 \vartheta}{dt^2} - \left(\frac{d\vartheta}{dt} \right)^2 + a^* \vartheta^2 \frac{d\vartheta}{dt} = 0, \quad (35)$$

which is easily integrated by substituting $d\vartheta/dt = p(\vartheta)$. Its solution has the form

$$\vartheta(t) = \vartheta_0 \left(1 - \frac{\vartheta^*}{\vartheta_0} \right) \left\{ 1 - \frac{\vartheta^*}{\vartheta_0} \exp [-a^* (\vartheta_0 - \vartheta^*) t] \right\}^{-1}. \quad (36)$$

If $\vartheta_0 < \vartheta^*$, then for $t > \tau$

$$\vartheta(t) = \vartheta_0 \left(1 - \frac{\vartheta^*}{\vartheta_0} \right) \exp (-t / \tau). \quad (37)$$

In the case where drop vaporization occurs in a compressible, highly turbulent gas flow, it can be shown that for the kinetic law $w = -L_y(y - y_s)$, $L_y = \mu \beta \rho_g \rho_Q^{-1}$

$$\frac{dy}{dt} + A(t) y_s(t) y(t) + A(t) y^2(t) = 0, \quad A = \frac{\mu \beta \rho_g}{\rho_Q r_0}, \quad (38), (39)$$

where y_s corresponds to that portion of the liquid phase at which the vapor in the vapor-gas mixture becomes saturated. Thus, the problem is complicated considerably, since additional relations are required for the functions $\rho_g(t)$ and $y_s(t)$. This can be done in principle, but it involves the use of numerical methods.

If a cloud of particles undergoes relative movement or is transported by a steady gas flow at a velocity $v(z)$ along a certain axis z , kinetic equation (1) takes the form

$$v \frac{\partial f}{\partial z} + \frac{\partial (fw)}{\partial r} = 0. \quad (40)$$

Literally repeating all of the preceding arguments, we may obtain a differential equation for $y(z)$

$$\frac{dy}{dz} - \frac{a^*v}{v} y = 0, \quad \frac{dy}{dz} = -a^* \frac{\vartheta}{v} \exp \left[-a^* \int_0^z \frac{\vartheta dz}{v(z)} \right] \quad (41)$$

and an equation for cooling of the gas

$$\frac{d\vartheta}{dz} = - \frac{a^* \vartheta^* \vartheta}{\left[1 + \frac{\mu c_{pv}}{c_{pg}} (1-y) \right] v} \exp \left[-a^* \int_0^z \frac{\vartheta dz}{v(z)} \right]. \quad (42)$$

Equations (27) and (29) again follow from (41) and (42). Allowing for (29), instead of (30) we may obtain from (41)

$$\frac{dy}{dz} + \frac{a^* (\vartheta_0 - \vartheta^*)}{v(z)} y + \frac{a^* \vartheta^*}{v(z)} y^2 = 0. \quad (43)$$

- Bernoulli's equation, which is easily integrated:

$$y = (\vartheta_0 - \vartheta^*) \left\{ \vartheta_0 \exp \left[a^* (\vartheta_0 - \vartheta^*) \int_0^z \frac{dz}{v(z)} \right] - \vartheta^* \right\}^{-1}. \quad (44)$$

Comparing (44) and (31), we see that if we make the substitution

$$t \rightarrow \int_0^z \frac{dz}{v(z)}$$

in Eqs. (31)-(35), then they can be used to calculate the vaporization of polydisperse particles in a steady gas flow. These equations also apply in the case of a cloud of drops moving relative to a stationary gas.

Let us present some sample values of the numerical constants in (13) and (24) for $n=0; 0.5; 1.0$: $C_1=1.00; 1.18; 1.25$; $C_1^{n-1}=1.0; 1.61; 1.56$; $k_R=3.0; 3.0; 2.38$. Thus, in cases of practical importance, $n=0.5$ and 1.0 , the value of ϑ^* in (24) is determined by the expression $\vartheta^* = \mu r_1 v / c_{pg}$. The criterional equation [2] $Nu = (2 + 0.3 \cdot Pr^{1/3} Re^{1/2})$ can be used to calculate α_0 , while the other quantities in (24) can be taken from handbook data.

NOTATION

r , drop radius; t , time; $f(r, t)$, drop-radius distribution function; $w = dr/dt$, vaporization rate; $N(t)$, N_0 , current and total number of particles in the system at the initial moment of time; A_i , a_i , constants in (4); $\alpha(r)$, α_0 , heat-transfer coefficients; ρ_q , ρ_g , density of liquid and gas; T_q , T , temperature of drop and gas; r_1 , heat of vaporization; $\vartheta = T - T_q$; n , exponent dependent on $\alpha(r)$; \bar{r}_0 , particle radius averaged over $f_0(r)$; y , x , fractions of unvaporized and vaporized moisture, respectively; $\Gamma(n)$, gamma function; ν , fraction of unvaporized particles at $t=t_0$; M_q , M_v , M_g , mass of drops, vapor, and gas, respectively; c_{pv} , c_{pg} , isobaric heat capacities of vapor and gas; a^* , ϑ^* , k_R , parameters defined in (9) and (24); μ , initial concentration of moisture (kg liquid/kg dry gas); τ , θ , characteristic times determined in (33) and (34); φ , function in the exponent of (26); β , mass-transfer coefficient; $v(z)$, particle velocity; z , coordinate. Indices: q , drops; v , vapor; g , gas; s , state of saturation; 0 , initial moment of time.

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MATHEMATICAL MODELING OF THE HEAT-TRANSFER
PROCESS AND SOLID PARTICLES IN A FLUIDIZED BED

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The authors formulate a two-concentration model of particle mixing in a fluidized bed, accounting for particle inertia.

In spite of the importance of knowing the laws for transfer of heat and particle mass in a fluidized bed, this matter has not yet been satisfactorily resolved. Existing methods account for some of the real transport properties of the system, but there is no model that describes at least the basic features of the bed and includes the present methods of describing the mixing phenomenon as special cases. For example, the simplest diffusion model [1] examines the particle transport process as a purely random diffusion one. The circulation mixing method [2] singles out convective particle transport as the basic mechanism (upwards in the trails of the ascending gas bubbles, and downwards in the remaining emulsion phase), and completely ignores the presence of diffusion transport in the continuous bed phase. The one-concentration equation of convective diffusion formulated in [3] takes account of both diffusion and circulation transport, and here to calculate the latter we need detailed knowledge of the particle velocity distribution over time and system volume. This makes the model extremely awkward for practical use. The circulation-diffusion two-concentration mixing method proposed in [4] combines the good qualities of the diffusion and the circulation models. But even it is not free from the common defect generally inherent in the diffusion parabolic equations, that it does not take account of the inertia of the solid phase. As is well known [5], this leads to the paradox of infinitely large instantaneous velocity of particle motion. Therefore, in [5] particle mixing was described by the hyperbolic diffusion equations which, however, do not account for the circulation mixing mechanism.

Thus, neither of the above models includes all the basic features of the mixing process, and therefore does not satisfactorily describe the actual process over a wide range of variation of the experimental conditions.

In this paper the authors have tried to construct quite a universal model of the process, to describe both diffusion and circulation transport, and the inertia of the solid phase.

Three basic mechanisms have been identified for mixing of particles and bubbles in fluidized beds (Fig. 1):

- a) circulation (convective) transport of particles vertically;
- b) turbulent diffusion of particles at finite speed in the descending dense phase;
- c) horizontal exchange of particles between trails of gas bubbles and the descending dense phase.

The continuity equations for flux of labeled particles have the form

$$A \frac{\partial c_1}{\partial \tau} + A \frac{\partial w_i c_1}{\partial x_i} = \beta (c_2 - c_1),$$